Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 1, pp. 65–70, 2007

MOTION OF AN INCLUSION IN UNIFORMLY AND NONUNIFORMLY VIBRATING LIQUIDS

V. L. Sennitskii

UDC 532.582

An approach to constructing the quantitative nonuniformity characteristics of liquid vibrations is proposed. A new problem of the motion of an inclusion in a vibrating liquid is considered.

Key words: *liquid, inclusion, uniform and nonuniform vibrations of a liquid.*

1. The motion of rigid and gas inclusions (bodies) in a vibrating liquid has been the subject of extensive experimental and theoretical research (see, for example, [1–18]). By a vibrating liquid is usually meant a liquid whose motion is caused by the action of vibrations.

The notions of uniformly and nonuniformly vibrating liquids were introduced in [8, 9] (see also [10]). In [8–10], uniform and nonuniform vibrations of a liquid are defined as follows. Let a liquid do not contain inclusions (whose motion is studied); then, the vibrations of the liquid are uniform if all particles of the liquid move at the same velocity and the vibrations of the liquid are nonuniform if not all particles of the liquid move with at the same velocity. The motion of inclusions in uniformly and nonuniformly vibrating liquids can differ qualitatively [2, 3, 8–10, 12, 13, 17, 18]. In view of this, the division of vibrations of liquids into uniform and nonuniform vibrations is the basis of their substantial classification. A natural extension of research in the direction outlined in [2, 3, 8, 12, 17, 18] involves investigating the questions of what parameters can be used for the quantitative characterization of the nonuniformity of liquid vibrations and what values of these parameters lead to qualitative changes in the motion of inclusions in vibrating liquids.

The present paper proposes a method for the quantitative characterization of the nonuniformity of liquid vibrations: the nonuniformity coefficient and the average nonuniformity coefficient of liquid vibrations are introduced. A new problem of the motion of a rigid inclusion in a vibrating liquid is formulated and solved. Conditions are found for the occurrence of the previously unknown state of the absence of mean motion of an inclusion in a vibrating liquid. It is established that the inclusion exhibits qualitatively different behavior in uniformly and nonuniformly vibrating liquids no matter how the nonuniformity of the liquid vibrations is slight.

2. We consider the following construction. There is a liquid which surrounds one or several bodies and is partially or completely surrounded by one or several walls. The liquid is vibrating: The liquid is set in motion by vibrations of the bodies and walls; the vibrations of the liquid at infinity are also possible.

We assume that Ω is the region in space occupied by the liquid, Γ is the boundary of the region Ω (including infinity if the region Ω is not bounded or is partially bounded externally), $\hat{V}_\Gamma$ is the largest magnitude of the liquid velocity on the boundary Γ, η is a point of space that belongs to the region Ω at any time, $\hat{V}_\eta$ is the largest magnitude of the liquid velocity at the point η, ω is the closed region of space that belongs to the region Ω at any time, and v is the volume of the region ω.

The nonuniformity coefficient of liquid vibrations at the point η and the average nonuniformity coefficient of liquid vibrations in the region ω are defined as

$$k_{\rm p}(\eta) = 1 - \hat{V}_\eta/\hat{V}_\Gamma; \tag{1}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; svovl@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 1, pp. 79–85, January–February, 2007. Original article submitted February 17, 2006.

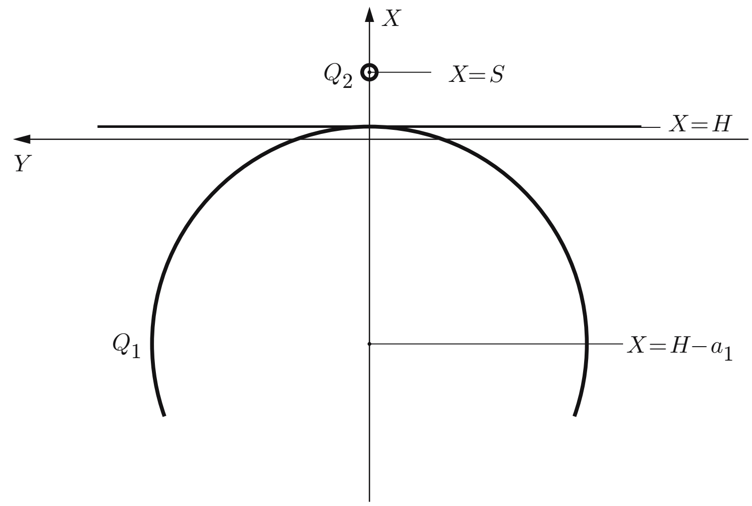

Fig. 1. System of coordinates, arrangement of the cylinder Q_1, the wall with the boundary permeable to the liquid, and the cylinder Q_2.

$$k_{\mathrm{reg}}(\omega) = \frac{1}{v} \iiint\limits_{\omega} k_{\mathrm{p}}\, d\omega. \tag{2}$$

We note that if the liquid performs uniform vibrations, which, for example, is the case in the problem considered in [2], then

$$k_{\mathrm{p}}(\eta) = 0, \qquad k_{\mathrm{reg}}(\omega) = 0$$

for any η and ω.

3. We consider the first auxiliary problem. There is an ideal incompressible liquid which is not bounded externally and contains an absolutely rigid, infinitely long circular cylinder Q_1 of radius a_1 (see Fig. 1). At the initial time $t = 0$, the cylinder and the liquid are at rest with respect to the inertial rectangular coordinate system X, Y, Z; the cylinder axis is parallel to the Z axis and passes through the point $(-a_1, 0, 0)$. At $t > 0$, the cylinder performs specified periodic (with period T) translational vibrations along the X axis; the position of the cylinder is defined by the radius-vector $(H - a_1)\boldsymbol{e}_X$ of the point of intersection of the cylinder axis with the X axis. Here

$$H = A(1 - \cos(2\pi t/T))$$

($A > 0$ is a constant) and $\boldsymbol{e}_X = (1, 0, 0)$. It is required to determine the plane potential liquid flow.

The liquid velocity potential φ_{c} obeys the equation

$$\Delta\varphi_{\mathrm{c}} = 0 \qquad (a_1 < R_1 < \infty) \tag{3}$$

and the conditions

$$\frac{\partial\varphi_{\mathrm{c}}}{\partial R_1} = \frac{dH}{dt}\cos\Theta_1 \qquad \text{at} \quad R_1 = a_1; \tag{4}$$

$$\nabla\varphi_{\mathrm{c}} \to 0 \qquad \text{at} \quad R_1 \to \infty; \tag{5}$$

$$\varphi_{\mathrm{c}} = 0 \qquad \text{at} \quad t = 0, \tag{6}$$

where R_1 and Θ_1 are polar coordinates in the plane $Z = 0$, which are related to X and Y by the equations

$$X = H - a_1 + R_1\cos\Theta_1, \qquad Y = R_1\sin\Theta_1.$$

Problem (3)–(6) has a solution

$$\varphi_{\mathrm{c}} = -a_1^2\frac{dH}{dt}\,\frac{X - H + a_1}{(X - H + a_1)^2 + Y^2}. \tag{7}$$

4. We consider the second auxiliary problem. There is an ideal incompressible liquid which is not bounded internally and is in contact with an absolutely rigid flat wall. The boundary of the wall is permeable to the liquid. At $t = 0$, the wall and the liquid are at rest with respect to the coordinate system X, Y, Z; the wall boundary coincides with the plane $X = 0$; the region occupied by the liquid beyond the wall is in the half-space $X \geq 0$. At $t > 0$, the wall performs specified periodic translational vibrations along the X axis; the position of the wall is defined by the radius-vector $H\boldsymbol{e}_X$ of the point of intersection of the wall boundary with the X axis; on the wall boundary, the liquid moves along the X axis at a velocity $(\partial\varphi_{\rm c}/\partial X)|_{X=H}$. It is required to determine the plane potential liquid flow beyond the wall.

The liquid velocity potential $\varphi_{\rm w}$ satisfies the equation

$$\Delta\varphi_{\rm w} = 0 \qquad (H < X < \infty) \tag{8}$$

and the conditions

$$\frac{\partial\varphi_{\rm w}}{\partial X} = \frac{\partial\varphi_{\rm w}}{\partial X} \qquad \text{at} \quad X = H; \tag{9}$$

$$\nabla\varphi_{\rm w} \to 0 \qquad \text{at} \quad X^2 + Y^2 \to \infty, \quad X \geq H. \tag{10}$$

Problem (8)–(10) has a solution

$$\varphi_{\rm w} = \varphi_{\rm c} + f, \tag{11}$$

where f is a function of t.

In the region $H < X < \infty$, the liquid is nonuniformly vibrating, according to (7) and (11).

Let $a_2 > 0$, $l > 0$, S_0 $(S_0 > 2A + a_2)$ be constants, η_0 be the point $(S_0, 0, 0)$, ω_0 be the closed region $[(X - S_0)^2 + Y^2]^{1/2} \leq a_2$, and $0 \leq Z \leq l$.

We assume that the values of $\delta = S_0/a_1$ and $\varepsilon = a_2/S_0$ are small compared to unity and the values of $\varkappa = A/a_2$ are not large compared to unity.

Using Eqs. (1), (2), (7), and (11) with accuracy up to quantities small compared to δ, we obtain

$$k_{\rm p}(\eta_0) = k_{\rm reg}(\omega_0) = k, \tag{12}$$

where

$$k = 2\delta. \tag{13}$$

According to (12) and (13), the quantities $k_{\rm p}(\eta_0)$ and $k_{\rm reg}(\omega_0)$, which coincide with each other in the indicated approximation, are the smaller and, hence, the liquid vibrations are closer to uniform vibrations, the larger a_1 and the smaller S_0.

5. Let us pass to the primal problem. There is an ideal incompressible liquid which contains an absolutely rigid, infinitely long, circular cylinder Q_2 of radius a_2 and is in contact with an absolutely rigid flat wall. The boundary of the wall is permeable to the liquid. At $t = 0$, the cylinder, the wall, and the liquid are at rest with respect to the coordinate system X, Y, Z; the cylinder axis is parallel to the Z axis and passes through the point η_0; the wall boundary coincides with the plane $X = 0$; the region occupied by the liquid beyond the wall is in the half-space $X \geq 0$. At $t > 0$, the wall performs specified periodic translational vibrations along the X axis; the position of the wall is defined by the radius-vector $H\boldsymbol{e}_X$ of the point of intersection of the wall boundary with the X axis; at the wall boundary, the liquid moves along the X axis at a velocity $(\partial\varphi_{\rm w}/\partial X)\Big|_{X=H}$; the liquid flow beyond the wall is plane and potential; the cylinder performs translational motion along the X axis under the action of the liquid pressure; the position of the cylinder is defined by the radius-vector

$$\boldsymbol{S} = S\boldsymbol{e}_X \tag{14}$$

of the point of intersection of the cylinder axis with the X axis ($S > H + a_2$). It is required to establish how the cylinder moves, i.e., it is required to determine the dependence of $\boldsymbol{S}$ on t.

We assume that Φ is the liquid velocity potential, P is the liquid pressure, R_2 and Θ_2 are polar coordinates in the plane $Z = 0$, which are related to X and Y by the equations

$$X = S + R_2 \cos\Theta_2, \qquad Y = R_2 \sin\Theta_2;$$

$$F = -a_2 l \int_0^{2\pi} P\Big|_{R_2=a_2} \cos\Theta_2 \, d\Theta_2 \tag{15}$$

is the force exerted by the liquid in the X direction on the part of the cylinder that occupies the region ω_0 at $t = 0$; ρ_{incl} is the density of the cylinder, ρ_{liq} is the density of the liquid, and I is a function of t.

The coordinate S, pressure P, and potential Φ satisfy the following equations and conditions:

$$\pi a_2^2 l \rho_{\text{incl}} \frac{d^2 S}{dt^2} = F; \tag{16}$$

$$S = S_0, \qquad \frac{dS}{dt} = 0 \qquad \text{at} \quad t = 0; \tag{17}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \frac{P}{\rho_{\text{liq}}} = I; \tag{18}$$

$$\Delta\Phi = 0; \tag{19}$$

$$\frac{\partial\Phi}{\partial X} = \frac{\partial\varphi_{\text{w}}}{\partial X} \qquad \text{at} \quad X = H; \tag{20}$$

$$\frac{\partial\Phi}{\partial R_2} = \frac{dS}{dt}\cos\Theta_2 \qquad \text{at} \quad R_2 = a_2; \tag{21}$$

$$\nabla\Phi \to 0 \quad \text{at} \quad X^2 + Y^2 \to \infty, \qquad X \geq H; \tag{22}$$

$$\Phi = 0 \qquad \text{at} \quad t = 0. \tag{23}$$

Problem(16)–(23) models the motion of a rigid inclusion — the cylinder Q_2 — in a nonuniformly vibrating liquid subjected to the vibrations from a rigid vibrator — the cylinder Q_1.

6. In (19)–(23), we make the substitution

$$X = x + H, \qquad Y = y, \qquad \Phi = \chi + \varphi_{\text{w}}. \tag{24}$$

As a result, we obtain the equation

$$\Delta\chi = 0 \tag{25}$$

and the conditions

$$\frac{\partial\chi}{\partial x} = 0 \qquad \text{at} \quad x = 0; \tag{26}$$

$$\frac{\partial\chi}{\partial r} = \frac{dS}{dt}\cos\theta - \frac{\partial\varphi_{\text{w}}}{\partial r} \qquad \text{at} \quad r = a_2; \tag{27}$$

$$\nabla\chi \to 0 \quad \text{at} \quad x^2 + y^2 \to \infty, \qquad x \geq 0; \tag{28}$$

$$\chi = c \qquad \text{at} \quad t = 0, \tag{29}$$

where r and θ are polar coordinates in the plane $Z = 0$, which are related to x and y by the equations

$$x = S - H + r\cos\theta, \qquad y = r\sin\theta;$$

$c = -\varphi_{\text{w}}|_{t=0}$ is a constant.

The values of $\lambda = \delta/\varepsilon^2$ are assumed to be not small and not large compared to unity.

Employing the method for determining the liquid velocity potential described in [12], we find the solution of problem (25)–(29) that satisfies Eqs. (25), (26), (28), and (29) exactly and satisfies Eq. (27) approximately, with accuracy up to quantities proportional to $a_2\, dH/dt$ and $a_2\, dS/dt$, which are small compared to $\varepsilon^3 a_2\, dH/dt$ and $\varepsilon^3 a_2\, dS/dt$, respectively. Using Eqs. (15), (18), and (24) and the indicated solution of problem (25)–(29), we obtain

$$F = \frac{\pi a_2^2 l \rho_{\rm liq} S_0}{T^2} \Big\{ 2\varepsilon \Big[1 + \frac{\varepsilon^2}{4s^2} \Big(1 + 2\varepsilon\, \frac{h}{s} \Big) - 2\lambda \varepsilon^2 s \Big(1 - \varepsilon\, \frac{h}{s} \Big) \Big] \frac{d^2 h}{d\tau^2} + \frac{\varepsilon^4}{2s^3} \Big(\frac{dh}{d\tau} \Big)^2$$
$$- \frac{\varepsilon^3}{s^3} \Big(1 + 3\varepsilon\, \frac{h}{s} \Big) \frac{dh}{d\tau}\, \frac{ds}{d\tau} - \Big[1 + \frac{\varepsilon^2}{2s^2} \Big(1 + 2\varepsilon\, \frac{h}{s} + 3\varepsilon^2\, \frac{h^2}{s^2} \Big) \Big] \frac{d^2 s}{d\tau^2}$$
$$+ \frac{\varepsilon^2}{2s^3} \Big(1 + 3\varepsilon\, \frac{h}{s} + 6\varepsilon^2\, \frac{h^2}{s^2} \Big) \Big(\frac{ds}{d\tau} \Big)^2 \Big\}, \tag{30}$$

where $\tau = t/T$, $h = H/a_2$, and $s = S/S_0$.

We assume that as $\varepsilon \to 0$

$$s \sim s_0 + \varepsilon s_1 + \ldots + \varepsilon^4 s_4. \tag{31}$$

Equations (16), (17), (30), and (31) imply the problems for $s_0, s_1, \ldots, s_4$. Solving these problems, we obtain

$$s_0 = 1, \quad s_1 = \tilde{s}_1, \quad s_2 = 0, \quad s_3 = \tilde{s}_3, \quad s_4 = -\frac{\pi^2}{2}\, \varkappa^2 \frac{\rho - 1}{(\rho + 1)^2} \Big(\frac{\rho - 1}{\rho + 1} + 8\lambda \Big) \tau^2 + \tilde{s}_4, \tag{32}$$

where $\rho = \rho_{\rm incl}/\rho_{\rm liq}$; $\tilde{s}_1, \tilde{s}_3, \tilde{s}_4$ are periodic functions of τ. Using (13), (31), and (32), we have

$$s = 1 + \Xi \tau^2 + \tilde{s}, \tag{33}$$

where

$$\Xi = -\frac{\pi^2}{2}\, \varepsilon^4 \varkappa^2 \frac{\rho - 1}{(\rho + 1)^2} \Big(\frac{\rho - 1}{\rho + 1} + \frac{4k}{\varepsilon^2} \Big); \tag{34}$$

$\tilde{s}$ is a periodic function of τ.

7. Formulas (14), (33), and (34) approximately define the dependence of $\boldsymbol{S}$ on t. The inclusion performs vibrations and the mean monotonic motion along the X axis.

Let us consider the mean motion of the inclusion.

According to (33) and (34), the following statements are valid.

1. If $\rho > 1$, then $\Xi < 0$ and the inclusion moves to the vibrator.

2. If $\rho = 1$, then $\Xi = 0$ and the inclusion does not perform mean motion.

3. If $0 \le \rho < 1$, then:

3.1. $\Xi < 0$ and the inclusion moves to the vibrator for $k = 0$ (uniform vibrations of the liquid) and for $0 < k < k^*$ (slight nonuniformity of the liquid vibrations).

3.2. $\Xi = 0$ and the inclusion does not perform mean motion for $k = k^*$.

3.3. $\Xi > 0$ and the inclusion moves from the vibrator for $k > k^*$ (strong nonuniformity of the liquid vibrations).

In statements 3.1–3.3,

$$k^* = \frac{\varepsilon^2}{4}\, \frac{1 - \rho}{1 + \rho}.$$

Let us make a qualitative comparison of the behavior (mean motion) of the inclusion in the primal problem of the present work with the behavior of the inclusion in the problems considered in [2], where the liquid vibrations are uniform, and in [3] where the liquid vibrations are nonuniform. Statements 1 and 2 are valid for both uniform and nonuniform vibrations of the liquid [2, 3]. Statement 3.1 agrees with the findings of [2]. Statement 3.3 agrees with findings of [3]. Thus, there is agreement with the results of [2, 3]. In addition, according to statement 3.2, there is a new, previously unknown, state of the absence of mean motion of the inclusion, which is the case for $0 \le \rho < 1$ and the "intermediate" nonuniformity of liquid vibrations where $k_{\rm p}(\eta_0) = k^*$ and $k_{\rm reg}(\omega_0) = k^*$. This state separates the other two states of qualitatively different behavior of the inclusion, in one of which the inclusion approaches the vibrator, and in the other, moves away from it.

Let

$$\rho^* = \frac{1 - 4k/\varepsilon^2}{1 + 4k/\varepsilon^2}$$

for $0 < k < \varepsilon^2/4$. According to (33) and (34), the following is the case:

1) For $\Xi < 0$, the inclusion moves to the vibrator if $0 \le \rho < \rho^*$;

2) For $\Xi > 0$, the inclusion moves from the vibrator if $\rho^* < \rho < 1$.

This result implies, in particular, that the motion of inclusions in uniformly and nonuniformly vibrating liquids can differ qualitatively no matter how the nonuniformity of liquid vibrations is slight.

REFERENCES

1. V. N. Chelomei, "Paradoxes in mechanics due to vibrations," *Dokl. Akad. Nauk SSSR*, **270**, No. 1, 62–67, (1983).
2. V. L. Sennitskii, "Motion of a circular cylinder in a vibrating liquid," *J. Appl. Mech. Tech. Phys.*, No. 5, 620–623 (1985).
3. V. L. Sennitskii, "Motion of a sphere in a liquid caused by vibrations of another sphere," *J. Appl. Mech. Tech. Phys.*, No. 4, 542–545 (1986).
4. B. A. Lugovtsov and V. L. Sennitskii, "Motion of a body in a vibrating liquid," *Dokl. Akad. Nauk SSSR*, **289**, No. 2, 314–317 (1986).
5. V. L. Sennitskii, "Motion of a gas bubble in a viscous vibrating liquid," *J. Appl. Mech. Tech. Phys.*, No. 6, 865–870 (1988).
6. V. L. Sennitskii, "Predominantly unidirectional motion of a gas bubble in a vibrating liquid," *Dokl. Akad. Nauk SSSR*, **319**, No. 1, 117–119 (1991).
7. V. L. Sennitskii, "Predominantly unidirectional motion of a compressible rigid body in a vibrating liquid," *J. Appl. Mech. Tech. Phys.*, No. 1, 96–97 (1993).
8. V. L. Sennitskii, "On motion of inclusions in uniformly and nonuniformly vibrating liquid," in: *Proc. of the Int. Workshop on G-Jitter*, Clarkson Univ., Potsdam (1993), pp. 178–186.
9. V. L. Sennitskii "Motion of inclusions in a vibrating liquid," Author's Abstract, Doct. Dissertation, Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1993).
10. V. L. Sennitskii, "Motion of inclusions in a vibrating liquid," *Sib. Fiz. Zh.*, No. 4, 18–26 (1995).
11. V. L. Sennitskii, "Behavior of a gas bubble in a viscous oscillating liquid in the presence of gravity," *J. Appl. Mech. Tech. Phys.*, **38**, No. 5, 718–723 (1997).
12. V. L. Sennitskii, "Motion of a sphere in a vibrating liquid in the presence of a wall," *J. Appl. Mech. Tech. Phys.*, **40**, No. 4, 662–668 (1999).
13. O. M. Lavrenteva, "On the motion of particles in nonuniformly vibrating liquid," *Europ. J. Appl. Math*, **10**, Part 3, 251–263 (1999).
14. V. L. Sennitskii, "Motion of a pulsing rigid body in an oscillating viscous fluid," *J. Appl. Mech. Tech. Phys.*, **42**, No. 1, 72–76 (2001).
15. I. E. Kareva and V. L. Sennitskii, "Motion of a circular cylinder in a vibrating liquid," *J. Appl. Mech. Tech. Phys.*, **42**, No. 2, 276–278 (2001).
16. V. L. Sennitskii, "Behavior of a pulsating rigid body in a viscous liquid in the presence of gravity," *J. Appl. Mech. Tech. Phys.*, **42**, No. 5, 814–817 (2001).
17. O. S. Pyatigorskaya and V. L. Sennitskii "Motion of a rigid body in a nonuniformly vibrating liquid," *Vestn. Novosib. Gos. Univ., Ser. Mat., Mekh., Inform.*, **2**, No. 2, 55–59 (2002).
18. O. S. Pyatigorskaya and V. L. Sennitskii, "Motion of a sphere in a liquid caused by vibrations of another sphere," *J. Appl. Mech. Tech. Phys.*, **45**, No. 4, 542–545 (2004).